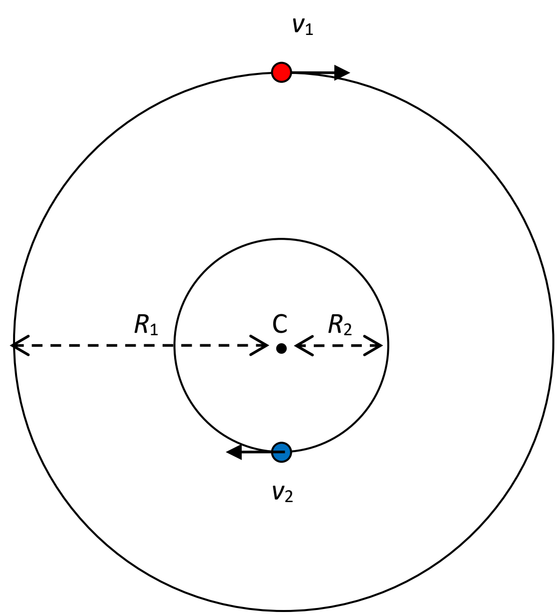
Teacher notes

Topic D

The binary star system

Two stars, of mass M_1 and M_2, orbit a common center C in circular orbits of radii R_1 and R_2, respectively. The stars are always diametrically opposite each other. This means they have the same angular speed and period.



(a) State and explain which is the faster star.

(b) Explain why the total momentum of the stars must be

(i) constant

(ii) why this constant momentum must in fact be zero.

(c) Hence deduce which is the more massive star.

(d) Deduce that $M_1R_1 = M_2R_2$.

(e) Hence show that $R_1 = \dfrac{M_2}{M_1+M_2}d$ and $R_2 = \dfrac{M_1}{M_1+M_2}d$ where $d = R_1 + R_2$ is the separation of the stars.

(f) Show that $v_1 = \sqrt{\dfrac{GM_2^2}{d(M_1+M_2)}}$ and $v_2 = \sqrt{\dfrac{GM_1^2}{d(M_1+M_2)}}$.

(g) Show that the common period of the stars is given by $T = 2\pi\sqrt{\frac{d^3}{G(M_1 + M_2)}}$.

(h) Simplify $M_1 v_1 R_1 + M_2 v_2 R_2$. Comment on the result.

(i) Verify that the total momentum of the two stars is zero.

(j) Show that the total energy of the two stars is $E_T = -\frac{1}{2}\frac{GM_1 M_2}{d}$.

The mass of each star is 1.4 times the mass of our Sun (2.0×10^{30} kg). The common period of the stars is 7.75 hours.

(k) Calculate
 (i) d,
 (ii) E_T.

The stars emit gravitational waves at a total rate of 7.35×10^{24} W.

(l) What is the loss of energy in 1 year?
(m) Using ideas of uncertainty propagation estimate
 (i) the change in the separation of the stars in 1 year,
 (ii) the change in the period of the stars in 1 year,
 (iii) the time for the two stars to run into each other.

Answers

(a) The outer star since it completes a larger distance in the same time.
(b)
 (i) The net external force on the system is zero.
 (ii) At any one time the stars are diametrically opposite so the net momentum is directed either along or opposite to one star's velocity. At some future time this direction would change and so momentum would change. But momentum is constant. The way to achieve this is if the total momentum is zero.
(c) Since $M_1 v_1 = M_2 v_2$ and $v_1 > v_2$ we deduce that $M_1 < M_2$.
(d) From $M_1 v_1 = M_2 v_2$ we deduce that $M_1 \omega R_1 = M_2 \omega R_2$ hence the result.
(e) $M_1 R_1 = M_2 R_2$ and $d = R_1 + R_2$. Solving the system of equations gives $R_2 = d - R_1$ and so $M_1 R_1 = M_2 (d - R_1)$ i.e. $M_1 R_1 + M_2 R_1 = M_2 d$ and finally $R_1 = \frac{M_2 d}{M_1 + M_2}$. Then $R_2 = \frac{M_1 d}{M_1 + M_2}$.

(f) $M_1\dfrac{v_1^2}{R_1}=\dfrac{GM_1M_2}{d^2}\Rightarrow v_1^2=\dfrac{GM_2R_1}{d^2}=\dfrac{GM_2}{d^2}\times\dfrac{M_2d}{M_1+M_2}=\dfrac{GM_2^2}{d(M_1+M_2)}$. Similarly for v_2.

(g) $T_1=\dfrac{2\pi R_1}{v_1}=2\pi\dfrac{\dfrac{M_2d}{M_1+M_2}}{\sqrt{\dfrac{GM_2^2}{d(M_1+M_2)}}}=2\pi\sqrt{\dfrac{d^3}{G(M_1+M_2)}}$. $T_2=\dfrac{2\pi R_2}{v_2}=2\pi\dfrac{\dfrac{M_1d}{M_1+M_2}}{\sqrt{\dfrac{GM_1^2}{d(M_1+M_2)}}}=2\pi\sqrt{\dfrac{d^3}{G(M_1+M_2)}}$.

This shows explicitly that the period is common.

(h) $M_1v_1R_1+M_2v_2R_2$. This is the total angular momentum of the system and so must be a constant.

$$M_1v_1R_1+M_2v_2R_2=M_1\sqrt{\frac{GM_2^2}{d(M_1+M_2)}}\frac{M_2d}{M_1+M_2}+M_2\sqrt{\frac{GM_1^2}{d(M_1+M_2)}}\frac{M_1d}{M_1+M_2}$$
$$=M_1M_2^2\sqrt{\frac{Gd}{(M_1+M_2)^3}}+M_2M_1^2\sqrt{\frac{Gd}{(M_1+M_2)^3}}$$
$$=M_1M_2(M_1+M_2)\sqrt{\frac{Gd}{(M_1+M_2)^3}}$$
$$=M_1M_2\sqrt{\frac{Gd}{M_1+M_2}}$$

This is a constant.

(i) We have that $M_1v_1=M_1\sqrt{\dfrac{GM_2^2}{d(M_1+M_2)}}=M_1M_2\sqrt{\dfrac{G}{(M_1+M_2)d}}$ and

$M_2v_2=M_2\sqrt{\dfrac{GM_1^2}{d(M_1+M_2)}}=M_1M_2\sqrt{\dfrac{G}{(M_1+M_2)d}}$ so the magnitudes are the same. The momenta are opposite so the total momentum is indeed zero.

(j) The total energy is:

$$E_T=\frac{1}{2}M_1v_1^2+\frac{1}{2}M_2v_2^2-\frac{GM_1M_2}{d}$$
$$=\frac{1}{2}M_1\frac{GM_2^2}{d(M_1+M_2)}+\frac{1}{2}M_2\frac{GM_1^2}{d(M_1+M_2)}-\frac{GM_1M_2}{d}$$
$$=\frac{1}{2}\frac{GM_1M_2^2}{d(M_1+M_2)}+\frac{1}{2}\frac{GM_2M_1^2}{d(M_1+M_2)}-\frac{GM_1M_2}{d}$$
$$=\frac{1}{2}\frac{GM_1M_2(M_1+M_2)}{d(M_1+M_2)}-\frac{GM_1M_2}{d}$$
$$=-\frac{1}{2}\frac{GM_1M_2}{d}$$

(k) (i) From $T = 2\pi\sqrt{\dfrac{d^3}{G(M_1+M_2)}}$ we find

$$d = \sqrt[3]{\frac{T^2G(M_1+M_2)}{4\pi^2}} = \sqrt[3]{\frac{(7.75\times 60\times 60)^2\times 6.67\times 10^{-11}\times(1.4\times 2.0\times 10^{30}\times 2)}{4\pi^2}}$$

$= 1.9\times 10^9$ m

(ii) $E_T = -\dfrac{1}{2}\dfrac{6.67\times 10^{-11}\times(1.4\times 2.0\times 10^{30})^2}{1.9\times 10^9} = -1.4\times 10^{41}$ J.

(l) In one year, the energy lost is $\Delta E = 7.35\times 10^{24}\times 365\times 24\times 60\times 60 = 2.3\times 10^{32}$ J.

(m) (i) $\dfrac{\Delta E}{E} = \dfrac{\Delta d}{d} \Rightarrow \Delta d = d\dfrac{\Delta E}{E} = 1.9\times 10^9\times\dfrac{2.3\times 10^{32}}{1.4\times 10^{41}} = 3.1 \approx 3\text{ m}$ per year.

(ii) $\dfrac{\Delta T}{T} = \dfrac{3}{2}\dfrac{\Delta d}{d} \Rightarrow \Delta T = \dfrac{3}{2}T\dfrac{\Delta d}{d} = \dfrac{3}{2}\times 7.75\times 60\times 60\times\dfrac{3.1}{1.9\times 10^9} \approx 68\ \mu\text{s}$ per year.

(iii) $T = \dfrac{1.4\times 10^{41}}{2.3\times 10^{32}} = 6.1\times 10^8$ years.

The numbers used are semi-realistic numbers for the Hulse-Taylor binary pulsar. For the measurement of the annual decrease in orbital period Hulse and Taylor received the 1993 Nobel Prize in Physics. They indirectly confirmed the existence of gravitational waves predicted by Einstein's theory of General Relativity.